

# Homological face-width condition forcing $K_6$ -minors in graphs on surfaces

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January 10, 2014

## Abstract

It is proved that every graph embedded on a (non-spherical) surface with non-separating face-width at least 7 contains a minor isomorphic to  $K_6$ . It is also shown that face-width four yields the same conclusion for graphs on the projective plane.

## 1 Introduction

A *surface* is a connected compact 2-manifold. Unless explicitly stated otherwise, surfaces will be assumed to be non-simply connected and have no boundary. If there is a nonempty boundary, then we speak of a *bordered surface* and every component of the boundary is called a *cuff*. A simple closed curve  $\gamma$  on a surface  $\Sigma$  is said to be *surface separating* or *zero-homologous* if cutting  $\Sigma$  along  $\gamma$  results in a disconnected (bordered) surface. Two disjoint simple closed curves are said to be *homologous* if they are either both zero-homologous, or none of them is zero-homologous, but cutting the surface along both of these curves disconnects the surface.

Let  $G$  be a graph embedded on a surface  $\Sigma$ . We regard  $G$  as a subset of  $\Sigma$  (that is, we identify  $G$  with its embedding on  $\Sigma$ ). The *face-width* of  $G$ , denoted by  $\text{fw}(G)$ , is the maximum number  $k$  so that every non-contractible simple closed curve in  $\Sigma$  intersects  $G$  in at least  $k$  points. The homology version, the *non-separating face-width* of  $G$ , denoted by  $\text{nsfw}(G)$ , is the maximum number  $k$  so that every surface non-separating simple closed curve

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\*Supported in part by postdoctoral support at the Simon Fraser University.

<sup>†</sup>Supported in part by an NSERC Discovery Grant (Canada), by the Canada Research Chair program, and by the Research Grant P1-0297 of ARRS (Slovenia).

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in  $\Sigma$  intersects  $G$  in at least  $k$  points. We refer to [10] for additional terminology involving graphs embedded in surfaces.

A graph  $H$  is a *minor* of a graph  $G$ , in symbols  $H \leq_m G$ , if  $H$  can be obtained from a subgraph of  $G$  by a series of contractions of edges.

The theory of graph minors (Robertson and Seymour [15]) shows that for every surface  $\Sigma$  there exists a constant  $c_\Sigma$  (depending only on  $\Sigma$ ) such that if  $G$  embeds in  $\Sigma$  with face-width at least  $c_\Sigma$ , then  $G$  contains  $K_6$  as a minor. We are interested in finding the best possible value for  $c_\Sigma$ . If  $G$  is an apex graph, then  $G$  does not contain  $K_6$  as a minor. It is known that there are apex graphs that can be embedded on non-spherical surfaces with face-width at least three, see [9]. Hence, there are surfaces  $\Sigma$  with  $c_\Sigma \geq 4$ . In fact, there are examples showing that  $c_\Sigma \geq 4$  for every surface  $\Sigma$ . We first show that  $c_\Sigma = 4$  in the special case when  $\Sigma$  is the projective plane.

**Theorem 1.1.** *Let  $G$  be a graph embedded on the projective plane. If  $\text{fw}(G) \geq 4$ , then  $K_6 \leq_m G$ .*

We suspect that  $c_\Sigma = 4$  for every  $\Sigma$ ; however a proof (or disproof) seems to be out of reach. Our main result given below extends Theorem 1.1 to arbitrary surfaces and strengthens the afore-mentioned result of Robertson and Seymour from [15] in two ways. First, we obtain an upper bound on  $c_\Sigma$  that is independent of the surface. In addition to this, we are able to loosen the face-width requirement by involving a condition on the non-separating face-width instead. Note that for graphs on the projective plane, we have  $\text{nsfw}(G) = \text{fw}(G)$  and that  $\text{nsfw}(G) \geq \text{fw}(G)$  holds in general.

**Theorem 1.2.** *Every graph  $G$  embedded on a non-spherical surface with  $\text{nsfw}(G) \geq 7$  contains the complete graph  $K_6$  as a minor.*

There is a continuing interest in the structure of graphs that do not contain  $K_6$  as a minor. An outstanding open problem in this area is a conjecture of Jørgensen [3] that every 6-connected graph has no  $K_6$ -minor if and only if it can be made planar by removing one vertex. An asymptotic version of Jørgensen’s Conjecture has been recently proved by Kawarabayashi et al. [7]. The known structure of such graphs is used in [5] in the design of an efficient algorithm for constructing linkless embeddings of graphs in 3-space. As for graphs embedded in surfaces, several papers [6, 11, 13, 12] concern  $K_6$ -minors in triangulations of surfaces of small genus, while [2] obtained a general result about  $K_6$ -minors in graphs in the projective plane.

All graphs in this paper are finite and simple. Paths and cycles have no “repeated vertices”. A path  $P = x_0x_1 \dots x_n$  is given by the sequence of its consecutive vertices  $x_0, x_1, \dots, x_n$ , but it is considered as a subgraph. If a path  $P$  has endvertices  $u$  and  $v$ , then  $P$  is called a  $(u, v)$ -path (also  $(v, u)$ -path). The *order* of a path  $P$ , denoted as  $|P|$ , is its number of vertices. For vertices  $a$  and  $b$  on a path  $P$ ,  $P[a, b]$  denotes the  $(a, b)$ -path contained in  $P$ , and  $P[a, b) = P[a, b] - b$  denotes the path from  $a$  to the predecessor of  $b$ . The paths  $P(a, b]$  and  $P(a, b)$  are defined analogously. The same notation is used for cycles with given clockwise orientation, where  $C[a, b]$  denotes the path from  $a$  to  $b$  in the clockwise direction.

For  $A_i \subseteq V(G)$  or  $A_i \subseteq G$  ( $i = 1, 2$ ), an  $(A_1, A_2)$ -path is an  $(a_1, a_2)$ -path  $P$  with  $V(P) \cap V(A_i) = \{a_i\}$  for  $i = 1, 2$ , an  $(A_1)$ -path is an  $(a_1, a_2)$ -path with  $V(P) \cap V(A_1) = \{a_1, a_2\}$ , where  $a_1 \neq a_2$  and  $P$  contains an edge that is not in  $A_1$ .

## 2 Face-chains

Let  $G$  be a graph embedded in a surface  $\Sigma$ . We denote by  $F(G)$  the set of all *facial walks* of  $G$ . Each facial walk is also considered as being a subgraph of  $G$  consisting of all vertices and edges on the boundary of a face of the embedding. The open face corresponding to the facial walk  $F$  will be denoted by  $F^\circ$ .

Let  $n \geq 0$  be an integer. A *face-chain*  $\Lambda$  of *length*  $n$  is an alternating sequence  $x_0, F_0, x_1, \dots, x_{n-1}, F_{n-1}, x_n$  such that, for  $i = 0, \dots, n-1$ ,  $F_i \in F(G)$  and  $x_i, x_{i+1} \in V(F_i) \cup E(F_i)$ . Note that  $x_{i+1}$  is either a vertex or an edge in  $F_i \cap F_{i+1}$ . We also write  $|\Lambda| = n$  to denote the length of  $\Lambda$ . If  $x_0 = x_n$ , then the face-chain is said to be *closed*. We define  $X(\Lambda) = \{x_0, \dots, x_n\}$  and  $G(\Lambda) = \bigcup_{i=0}^{n-1} F_i \subseteq G$ .

Let  $\Lambda = x_0, F_0, x_1, \dots, x_{n-1}, F_{n-1}, x_0$  be a closed face-chain. We define a closed curve  $\Gamma(\Lambda) \subseteq \Sigma$  by taking the composition of simple arcs in each  $F_i$  joining  $x_i$  and  $x_{i+1}$ . (Note that the choice of a simple arc in  $F_i$  is determined up to homotopy, if we assume that every  $F_i$  is homeomorphic to an open disk and that each of  $x_i$  and  $x_{i+1}$  appears in the facial walk  $F_i$  just once; these assumptions will always be satisfied.)

We say that a face-chain  $\Lambda$  is *nice*, if for all  $0 \leq i < j \leq n-1$ , we have  $F_i \neq F_j$ ,  $x_i \neq x_j$ , and  $x_k \neq x_n$  for  $1 \leq k < n$ . Note that if  $\Lambda$  is nice then  $\Gamma(\Lambda)$  is simple. A nice face-chain  $\Lambda$  is *clean* if for  $i = 0, \dots, n-1$ ,  $F_i \cap F_{i+1} = \{x_{i+1}\}$  (where  $F_n = F_0$ ) and for all  $0 \leq i < j \leq n-1$ , with  $1 \neq j - i \neq n-1$ , we have  $F_i \cap F_j = \emptyset$ .

To avoid repetition let us state the following assumption together with its notation, since it will be common to several statements that follow.

**(H1)** For a graph  $G$  embedded in a surface  $\Sigma$ , let  $\Lambda = x_0, F_0, x_1, \dots, F_{n-1}, x_0$  be a closed face-chain of length  $n$  such that:

- (i)  $\Gamma(\Lambda)$  is surface non-separating;
- (ii) subject to (i),  $|\Lambda|$  is minimum.

We shall abuse terminology and call a face-chain  $\Lambda$  surface separating or contractible when  $\Gamma(\Lambda)$  has that property.

The following result is well-known (cf. [10]) and is referred to as the *3-path condition*.

**Theorem 2.1.** *Let  $G$  be a graph embedded on  $\Sigma$ , and let  $x, y \in V(G)$ . Suppose  $G$  contains three  $(x, y)$ -paths,  $P_1, P_2, P_3$ , pairwise disjoint except for their ends. Let  $C_{ij}$  ( $1 \leq i < j \leq 3$ ) be the cycle  $P_i \cup P_j$ . Then the following holds:*

- (a) *If two of the three cycles  $C_{ij}$  are contractible, then so is the third.*

(b) *If two of the three cycles  $C_{ij}$  are surface separating, then so is the third.*

Let  $\Lambda$  be a closed face-chain. Let  $\Lambda' = w_0, F'_1, w_1, \dots, w_{k-1}, F'_k, w_k$  be a face-chain (not closed) of length  $k$  such that  $w_0$  is incident with a face  $F_i$  and  $w_k$  is incident with a face  $F_j$  ( $0 \leq i < j \leq n-1$ ) in the face-chain  $\Lambda$ . There are two face-chains in  $\Lambda$  whose first and last faces are  $F_i$  and  $F_j$ . We can combine each of these with  $\Lambda'$  to get a closed face-chain containing  $\Lambda'$ . By using the 3-path property, we deduce the following.

**Theorem 2.2.** *Let  $G$  be a graph embedded in  $\Sigma$ , and let  $\Lambda$  be as in (H1). Let  $\Lambda' = w_0, F'_1, w_1, \dots, w_{k-1}, F'_k, w_k$  be a face chain of length  $k \geq 0$ , where  $w_0$  and  $w_k$  vertices or edges that are incident with faces  $F_i$  and  $F_j$  in  $\Lambda$  ( $0 \leq i < j \leq n-1$ ). Then the closed face-chain formed by  $\Lambda'$  and the shorter one of the two face-chains from  $F_i$  to  $F_j$  in  $\Lambda$  is a face-chain of length  $\leq 2k+2$ .*

*Proof.* Let  $\Lambda_1 = F_i, x_{i+1}, F_{i+1}, \dots, x_j, F_j$  and  $\Lambda_2 = F_j, x_{j+1}, F_{j+1}, \dots, x_0, F_0, \dots, x_i, F_i$  be the two face-subchains from  $F_i$  to  $F_j$  contained in  $\Lambda$ . They together use  $n+2$  faces. Let us now consider the two closed face-chains  $\Lambda_1 \cup \Lambda'$  and  $\Lambda_2 \cup \Lambda'$ . Clearly,

$$|\Lambda_1 \cup \Lambda'| + |\Lambda_2 \cup \Lambda'| = n + 2 + 2k.$$

By the 3-path condition, at least one of them is surface non-separating, thus it is of length at least  $n$  by (H1)(ii). So, it follows that the length of the other one is at most  $2k+2$ .  $\square$

The following theorem is well-known for the face-width (cf. [10]); the proof for non-separating face-width is essentially the same.

**Theorem 2.3.** *Let  $G$  be a 3-connected graph embedded on a surface  $\Sigma$  with  $\text{nsfw}(G) \geq 3$ . Then all facial walks of  $G$  are cycles, and any two of them are either disjoint or intersect in a single vertex or a single edge.*

The following is an easy corollary of Theorem 2.3, Theorem 2.2 (with  $k=0$ ) and the 3-path condition.

**Theorem 2.4.** *Let  $G$  be a 3-connected graph embedded on a surface  $\Sigma$  with  $\text{nsfw}(G) \geq 3$ , and let  $\Lambda$  be as in (H1). Then  $\Lambda$  is clean.*

The following result is an easy corollary of the 3-path condition. Its proof for the edge-width can be found in [16]; for the proof of the face-width version, see [10]; the proof for the non-separating face-width is essentially the same as in [10].

**Theorem 2.5.** *Let  $G$  be embedded in a surface  $\Sigma$ , and let  $\Lambda$  be as in (H1). Let  $G'$  be obtained from  $G$  by cutting  $\Sigma$  along  $\Gamma(\Lambda)$  and capping off the resulting cuffs. Then  $\text{fw}(G') \geq \lceil \frac{1}{2} \text{fw}(G) \rceil$  and  $\text{nsfw}(G') \geq \lceil \frac{1}{2} \text{nsfw}(G) \rceil$ .*

Let  $G$  be a graph embedded on  $\Sigma$  and let  $p \in \Sigma \setminus G$  be a preselected point on the surface. If  $C$  is a surface-separating cycle of  $G$ , we denote by  $\text{Int}(C)$  the subgraph of  $G$  contained in the part of the surface separated by  $C$  that contains  $p$ ; in particular,  $C \subseteq \text{Int}(C)$ . Let

$f \in F(G)$  be a face of  $G$ . We define subgraphs  $B_0(f), B_1(f), B_2(f), \dots$  of  $G$  recursively as follows:  $B_0(f) = f$ , and for  $k \geq 1$ ,  $B_k(f)$  is the union of  $B_{k-1}(f)$  and all facial walks that have a vertex in  $B_{k-1}(f)$ . Let  $\partial B_k(f)$  be the set of edges of  $B_k(f)$  (together with their ends) that are not incident with a vertex of  $B_{k-1}(f)$ . With this notation we have the following result (see [8]).

**Theorem 2.6.** *Let  $G$  be a graph embedded on  $\Sigma$  with  $\text{nsfw}(G) \geq 2$ . Let  $f \in F(G)$  and let  $k = \lfloor \frac{1}{2} \text{nsfw}(G) \rfloor - 1$ . Then there exist pairwise disjoint surface-separating cycles  $C_0(f), \dots, C_k(f)$  such that for  $i = 0, \dots, k$ ,  $C_i(f) \subseteq \partial B_i(f)$  and  $B_i(f) \subseteq \text{Int}(C_i(f))$  (where  $\text{Int}$  is defined with respect to a point  $p$  in the face  $f$ ). Moreover, if  $l = \lfloor \frac{1}{2} \text{fw}(G) \rfloor - 1$ , then the cycles  $C_0, C_1, \dots, C_l$  are contractible in  $\Sigma$ .*

In addition to having large  $\text{nsfw}(G)$ , we will also need  $\text{fw}(G)$  to be large. This will be made possible by the following theorem.

**Theorem 2.7.** *Let  $G$  be a graph embedded in a surface  $\Sigma$  with  $k = \text{nsfw}(G) \geq 6$ . Then  $G$  contains a minor  $G'$  such that  $G'$  is 3-connected and has an embedding in a surface  $\Sigma'$  with  $\text{nsfw}(G') = k$  and  $\text{fw}(G') \geq 6$ .*

*Proof.* Let  $G'$  be a minor of  $G$  with the minimum number of vertices and edges such that  $G'$  has an embedding in a surface  $\Sigma'$  with  $\text{nsfw}(G') = k$ . Clearly,  $G'$  exists. We claim that  $\text{fw}(G') \geq 6$ . If not, let  $1 \leq l \leq 5$  be the smallest integer such that there exists a closed face-chain  $\Lambda = x_0, F_0, x_1, \dots, F_{l-1}, x_0$  with  $\Gamma(\Lambda)$  non-contractible. Since  $k \geq 6$ ,  $\Gamma(\Lambda)$  is surface-separating. Let  $\Sigma'_1$  and  $\Sigma'_2$  be the surfaces obtained by cutting  $\Sigma'$  along  $\Gamma(\Lambda)$  and capping off the resulting cuff. For  $i = 1, 2$ , let  $G'_i$  be the subgraph of  $G'$  in  $\Sigma'_i$ , and let  $G''_i$  be obtained from  $G'_i$  by adding a vertex of degree  $l$  and joining it to all vertices in  $X = \{x_0, \dots, x_{l-1}\}$  (and embedding the vertex and these edges into the capped disk). Each face  $F_j$  ( $0 \leq j < l$ ) determines a face  $F_j^i$  in  $\Sigma'_i$ . This correspondence makes it possible to convert every face-chain in  $G''_1$  to a face-chain in  $G'$ . Note that  $G''_1$  is a proper minor of  $G'$  (since  $l$  is smallest,  $G'_2 \setminus X$  contains a connected component adjacent to all vertices in  $X$  and can thus be contracted into the added vertex of  $G''_1$ ). By the minimality of  $G'$ , we conclude that the embedding of  $G''_1$  in  $\Sigma'_1$  has  $\text{nsfw}(G''_1) < k$ . Let  $\Lambda'$  be a non-separating closed face-chain of length  $k' < k$  confirming this fact. It is easy to see that  $\Lambda'$  determines a non-separating face-chain in  $G'$  of the same length (since  $l \leq 5$ ). This contradiction proves that  $\text{fw}(G') \geq 6$ .

Finally, since  $\text{fw}(G') \geq 3$ ,  $G'$  contains a 3-connected minor whose face-width and non-separating face-width are the same ([10]). By the minimality of  $G'$ , this minor is equal to  $G'$ . This completes the proof.  $\square$

### 3 Disjoint paths on a surface

Let  $G$  be a graph embedded on  $\Sigma$ . Let  $C_1, C_2 \subseteq G$  be disjoint, homologous, surface non-separating cycles in  $G$ . Note that  $C_1$  and  $C_2$  are 2-sided since pairs of 1-sided homologous cycles always intersect each other. Let  $\Sigma_0$  and  $\Sigma_1$  be bordered surfaces, whose cuffs coincide

with  $C_1$  and  $C_2$ , where  $\Sigma_0 \cup \Sigma_1 = \Sigma$ ,  $\Sigma_0 \cap \Sigma_1 = C_1 \cup C_2$ . Similarly, we can write  $G = G_0 \cup G_1$ , where  $G_i$  is the subgraph of  $G$  embedded in  $\Sigma_i$  for  $i = 0, 1$ , and thus  $G_0 \cap G_1 = C_1 \cup C_2$ . For  $i = 0, 1$ , we denote by  $\Sigma'_i$  the closed surface obtained from the bordered surface  $\Sigma_i$  by capping off the two cuffs of  $\Sigma_i$ . With this notation we have the following:

**Lemma 3.1.** *Each of  $G_0$  and  $G_1$  contains  $\text{nsfw}(G)$  pairwise disjoint  $(C_1, C_2)$ -paths.*

For the proof of Lemma 3.1, we need the following result whose weaker form for contractible curves has appeared in [1].

**Lemma 3.2.** *Let  $G$  be a graph embedded on a surface  $\Sigma$ , and let  $A$  (possibly  $A = \emptyset$ ) be a set of vertices such that  $G' = G - A$  is disconnected. Let  $\hat{C}_1$  and  $\hat{C}_2$  be distinct connected components of  $G'$ . Then  $\Sigma$  contains a simple closed curve  $\Gamma$  such that  $\Gamma \cap G \subseteq A$  and if  $\Gamma$  is surface separating, then  $\hat{C}_1$  and  $\hat{C}_2$  are contained in different connected components of  $\Sigma \setminus \Gamma$ .*

*Proof.* Consider the disconnected graph  $G'$  with its induced embedding on  $\Sigma$ . We claim that

- (1)  $\Sigma \setminus G'$  contains a 2-sided simple closed curve  $\Gamma$  that intersects  $G$  only in edges joining  $\hat{C}_1$  with  $A$ , and  $\Gamma$  is either surface non-separating in  $\Sigma$ , or separates  $\Sigma$  into two components, one containing  $\hat{C}_1$  and the other one containing  $\hat{C}_2$ .

To see this, let us first delete all components of  $G'$  distinct from  $\hat{C}_1$  and  $\hat{C}_2$ . Next, let us add an edge  $e$  joining a vertex in  $\hat{C}_1$  with a vertex in  $\hat{C}_2$  so that the resulting graph  $G'' = \hat{C}_1 \cup \hat{C}_2 + e$  is embedded in  $\Sigma$ . Since  $e$  is a cut-edge of  $G''$ , the unique facial walk  $F$  containing  $e$  in the induced embedding of  $G''$  contains  $e$  twice and  $e$  is traversed in opposite directions. Following the part of this facial walk in  $\hat{C}_1$ , we see that  $\Sigma$  contains a simple closed curve  $\Gamma$  that follows the boundary of  $F$  close to  $\hat{C}_1$  so that  $\Gamma$  crosses  $e$  exactly once, and  $\Gamma$  intersects only  $e$  and the edges of  $G$  joining  $\hat{C}_1$  with  $A$ . In particular,  $\Gamma$  does not intersect any of the removed components of  $G'$ . If  $\Gamma$  separates  $\Sigma$ , then each component of  $\Sigma \setminus \Gamma$  contains exactly one of the components  $\hat{C}_1$  or  $\hat{C}_2$  since the edge  $e$  crosses  $\Gamma$ . This proves (1).

Let us consider all simple closed curves satisfying the conclusion of (1), except that we allow them to intersect  $G$  not only at interior points of the edges joining  $\hat{C}_1$  with  $A$ , but also allow that  $\Gamma$  passes through vertices in  $A$ . Among all such curves, choose  $\Gamma \subseteq \Sigma$  having minimum number of crossings with interior points on the edges joining  $\hat{C}_1$  with  $A$ . Note that  $\Gamma$  intersects  $G$  only in  $A$  or in edges joining  $A$  to vertices in  $\hat{C}_1$ . By possibly altering  $\Gamma$ , we may assume that each intersection of  $\Gamma$  with an edge of  $G$  is a crossing.

If  $\Gamma \cap E(G) = \emptyset$ , then  $\Gamma$  is of the desired form and the claim follows. Hence  $\Gamma$  intersects an edge  $a = uv \in E(G)$ , where  $u \in V(\hat{C}_1)$  and  $v \in A$ . Replace a short segment of  $\Gamma$  around this intersection with a simple curve which follows  $a$  to its endvertex  $v$  in  $A$ , crosses through  $v$  and returns back on the other side of  $a$  (if  $\Gamma$  intersects  $a$  in more than one point, choose the intersection point which is closest to  $v$ ). The resulting curve  $\Gamma'$  is homotopic to  $\Gamma$ . By the minimality property of  $\Gamma$ ,  $\Gamma'$  is not simple, and is hence composed of two simple closed curves  $\Gamma_1$  and  $\Gamma_2$  that intersect at  $v$ .

We may assume that both  $\Gamma_1$  and  $\Gamma_2$  separate  $\Sigma$ , for if  $\Gamma_i$  ( $1 \leq i \leq 2$ ) does not separate  $\Sigma$ , then  $\Gamma_i$  can be chosen instead of  $\Gamma$ , contradicting the minimality of  $\Gamma$ . By our choice of  $\Gamma$  and  $a$ , it is easy to see that there must exist  $i \in \{1, 2\}$  such that cutting  $\Sigma$  along  $\Gamma_i$  disconnects  $\Sigma$  into two components each containing exactly one component  $\hat{C}_1$  or  $\hat{C}_2$ . But then  $\Gamma_i$  can be chosen instead of  $\Gamma$ , contradicting the minimality of  $\Gamma$ . This completes the proof.  $\square$

*Proof of Lemma 3.1.* By symmetry, it suffices to prove the lemma for  $G_1$ . Let  $r$  be the maximum number of disjoint  $(C_1, C_2)$ -paths contained in  $G_1$ . To prove the claim, we have to show that  $r \geq \text{nsfw}(G)$ . By Menger's theorem there exists  $A \subseteq V(G_1)$  with  $|A| = r$  that separates  $C_1$  and  $C_2$ . Let  $G_2 \supseteq G_1$  be the graph embedded in  $\Sigma'_1$  that is obtained from  $G_1$  by adding two vertices  $v_1, v_2$ , where  $v_i$  is adjacent to all vertices in  $C_i$  ( $i = 1, 2$ ). For  $i = 1, 2$ , let  $\hat{C}_i$  be the connected component of  $G_2 - A$  containing  $v_i$ . Then  $A$  satisfies assumptions of Lemma 3.2. Let  $\Gamma$  be a simple closed curve on  $\Sigma'_1$  as promised to exist by Lemma 3.2. If  $\Gamma$  is surface-separating in  $\Sigma'_1$ , then it separates  $v_1$  from  $v_2$ . Moreover,  $\Gamma \cap G_2 \subseteq A$  and we may assume that  $\Gamma$  is disjoint from the interior of  $\Sigma_0$ . However, in the surface  $\Sigma$ ,  $\Gamma$  is surface non-separating since the two components of  $\Sigma'_1 \setminus \Gamma$  are connected together in  $\Sigma \setminus \Gamma$  through  $\Sigma_0$ . Thus, we conclude that  $\Gamma$  is always surface non-separating. Therefore,  $r \geq |\Gamma \cap G| = |\Gamma \cap G_2| \geq \text{nsfw}(G)$ .  $\square$

For a path  $P$ , we denote by  $\text{int}(P)$  the path obtained from  $P$  by removing its end-vertices (and incident edges).

**Theorem 3.3.** *Let  $G, C_1, C_2$  and  $G_1, \Sigma_0, \Sigma_1$  be as introduced at the beginning of the section. Let  $\mathcal{P}$  be a set of pairwise disjoint  $(C_1, C_2)$ -paths in  $G_1$  of maximum cardinality. Let  $i \in \{1, 2\}$ , and let  $w, w' \in V(C_i)$  be two vertices of  $C_i$ . Let  $X, \bar{X} \subseteq C_i$  be the two  $(w, w')$ -paths on  $C_i$ , i.e.,  $C_i = X \cup \bar{X}$  and  $X \cap \bar{X} = \{w, w'\}$ . Then one of the following holds:*

- (a) *There exists an  $(\text{int}(\bar{X}), \mathcal{P})$ -path in  $G_1$  disjoint from  $X$ . (Here we consider  $\mathcal{P}$  as a subgraph of  $G$ .)*
- (b) *There exist  $v, u \in V(X)$  such that  $v$  and  $u$  are incident with a common face  $f \in F(G_1)$  in  $\Sigma_1$ , and the closed curve in  $\Sigma$  formed by a simple arc in  $f^\circ$  from  $u$  to  $v$  together with the segment  $X[v, u]$  on  $X$  is surface non-separating in  $\Sigma$ .*

*Proof.* By symmetry we may assume that  $i = 1$ . We may also assume that  $\text{int}(\bar{X}) \neq \emptyset$  since otherwise (b) holds with  $\{v, u\} = \{w, w'\}$ . Moreover, no path in  $\mathcal{P}$  has an end in  $\text{int}(\bar{X})$  since otherwise (a) holds. We will assume that (a) fails, and show that (b) holds. In particular, we will show that there exists a simple arc  $\gamma$  in  $\Sigma_1$ , so that  $\gamma \cap G_1 = \{v, u\}$ , where  $v, u \in V(X)$ , and  $\gamma \cup X[v, u]$  is a surface non-separating closed curve. This will imply (b).

So, suppose (a) does not hold. The maximality of  $|\mathcal{P}|$  and the assumption that (a) does not hold, imply that in  $G_1 - V(X)$ , there is no  $(\text{int}(\bar{X}), \mathcal{P} \cup C_2)$ -path. Hence,  $G_1 - V(X)$  is disconnected, with  $C_2$  and  $\text{int}(\bar{X})$  belonging to distinct connected components.

Let  $G_2$  (embedded on  $\Sigma'_1$ ) be obtained from  $G_1 - V(X)$  by adding an edge  $e$  (embedded along the deleted path  $X$ , but drawn inside the capped face in  $\Sigma'_1$  so that it does not intersect

$G_1$ ) connecting the two end vertices of  $\text{int}(\overline{X})$ . Note that  $G_2$  has the same connected components as  $G_1 - V(X)$ , since  $e$  connects two vertices of  $\text{int}(\overline{X})$  that are in the same component of  $G_1 - V(X)$ . Let  $F_1$  be the face of  $G_2$  in  $\Sigma'_1$  bounded by the cycle  $C'_1 = \text{int}(\overline{X}) + e$ .

Clearly,  $C'_1$  is a cycle in  $G_2$  which is homotopic to  $C_1$  in  $\Sigma$ . Let  $\hat{C}'_1$  and  $\hat{C}_2$  be the connected components of  $G_2$  containing  $C'_1$  and  $C_2$ , respectively. Let  $\Gamma \subseteq \Sigma'_1$  be the closed curve obtained by applying Lemma 3.2 to the embedded graph  $G_1 + e \subseteq \Sigma'_1$ , the separating vertex set  $V(X)$  playing the role of  $A$ , and considering the connected components  $\hat{C}'_1$  and  $\hat{C}_2$  of  $G_2 = (G_1 + e) - V(X)$ . Then  $\Gamma \cap \text{int}(F_1) = \emptyset$  since  $\Gamma \cap \hat{C}'_1 = \emptyset$ . In particular,  $\Gamma \subseteq \Sigma_1$  and  $\Gamma \cap G_1 \subseteq V(X)$ . We claim that

- (1)  $\Gamma$  is surface non-separating in  $\Sigma$ .

This is clear if  $\Gamma$  is surface non-separating in  $\Sigma'_1$ . Otherwise,  $\Gamma$  is surface separating in  $\Sigma'_1$ . As guaranteed by the use of Lemma 3.2,  $\Gamma$  separates  $\hat{C}'_1$  from  $\hat{C}_2$  in  $\Sigma'_1$ . However, in  $\Sigma$ , these two parts are connected together via the surface part  $\Sigma_0$ , so  $\Gamma$  is not surface separating in  $\Sigma$ . This proves (1).

In the sequel we will consider curves in  $\Sigma'_1 \setminus \text{int}(F_1)$ . We can view  $\Sigma_1 \subset \Sigma'_1 \setminus \text{int}(F_1) \subset \Sigma$  and therefore talk about homology properties of such curves in  $\Sigma$ .

Let  $\Gamma_1$  be a closed curve in  $\Sigma'_1 \setminus \text{int}(F_1)$  so that the following conditions hold:

- (i)  $\Gamma_1 \cap (G_1 + e) \subseteq V(X)$ ,  $\Gamma_1$  is surface non-separating in  $\Sigma$ , and every arc  $\gamma \subseteq \Gamma$  with ends  $x, y \in V(X)$  and which is otherwise disjoint from  $\Sigma_1$  is homotopic to  $X[x, y]$ ;
- (ii) subject to (i), the number of connected components of  $\Gamma_1 \cap \Sigma_1$  is minimum.

Note that such a choice of  $\Gamma_1$  is possible since  $\Gamma$  satisfies (i).

The curve,  $\Gamma_1$  is surface non-separating in  $\Sigma$ . By (i) we deduce that there exists an arc  $\gamma \subseteq \Gamma_1$  contained in  $\Sigma_1$  such that  $\gamma \cap G_1 = \{x, y\}$  where  $x, y \in V(X)$ . Let  $\gamma'$  be a curve in  $\Sigma'_1 \setminus \Sigma_1$  along  $X$  with ends  $x$  and  $y$  that is homotopic to  $X[x, y]$ .

If  $\gamma \cup \gamma'$  is surface non-separating in  $\Sigma$ , then (b) holds. Otherwise, by replacing  $\gamma$  in  $\Gamma_1$  with  $\gamma'$ , we obtain a new curve  $\Gamma_2$  that satisfies (i), by the 3-path condition. But then the existence of  $\Gamma_2$  contradicts (ii) in the choice of  $\Gamma_1$ . This contradiction concludes the proof.  $\square$

The following result is a well-known corollary of Menger's theorem.

**Lemma 3.4.** *Let  $\Sigma$  be a cylinder and let  $F_1$  and  $F_2$  be the two cuffs. Let  $G$  be a graph embedded on  $\Sigma$  and suppose that for  $i = 1, 2$ ,  $S_i := F_i \cap G \subseteq V(G)$ . Let  $r \geq 0$  be an integer. Suppose that every simple closed curve  $\Gamma$  with  $\Gamma \cap G \subseteq V(G)$  and  $|\Gamma \cap V(G)| < r$  is contractible in  $\Sigma$ . Then there are  $r$  pairwise disjoint  $(S_1, S_2)$ -paths in  $G$ .*



## 4 A grid on a cylinder

Let  $C$  be a cycle and let  $S \subseteq V(C)$  be a subset of its vertices. For  $x, y \in V(C)$ , let  $A$  and  $B$  be the two components (possibly empty) of  $C - \{x, y\}$ . We define the *distance* between  $x$  and  $y$  on  $C$  with respect to  $S$ , denoted by  $\text{dist}_{(C,S)}(x, y)$  to be  $\min\{|V(A) \cap S|, |V(B) \cap S|\}$ .

**Theorem 4.1.** *Let  $G$  be a graph embedded in a cylinder, and suppose that  $G$  has three pairwise disjoint homotopic cycles  $C_1, C_2, C_3$ , such that  $C_1$  and  $C_3$  coincide with the cuffs of the cylinder. Let  $k \geq 7$  be an integer and let  $P_0, \dots, P_{k-1}$  be pairwise disjoint  $(C_1, C_3)$ -paths in  $G$  such that for every  $0 \leq i \leq k-1$  and  $1 \leq j \leq 3$ , the intersection of  $P_i$  and  $C_j$  is a single vertex. For  $i = 0, \dots, k-1$ , let  $s_i$  and  $t_i$  be the ends of  $P_i$  on  $C_1$  and  $C_3$ , respectively. Set  $S := \{s_0, \dots, s_{k-1}\}$  and  $T := \{t_0, \dots, t_{k-1}\}$ . Let  $a_1, a_2 \in V(C_1)$  and  $b_1, b_2 \in V(C_3)$  such that  $b_1 \neq b_2$  and  $\text{dist}_{(C_1,S)}(a_1, a_2) \geq 2$ . Then the following holds:*

- (i) *If  $\text{dist}_{(C_3,T)}(b_1, b_2) \geq 1$ , then  $G' = G + a_1b_1 + a_2b_2$  contains a  $K_6$ -minor.*
- (ii) *If  $\text{dist}_{(C_3,T)}(b_1, b_2) = 0$ , and there exist vertices  $a_3 \in V(C_1)$  and  $b_3 \in V(C_3)$ , such that  $\text{dist}_{(C_3,T)}(b_1, b_3) \geq 1$  or  $\text{dist}_{(C_3,T)}(b_2, b_3) \geq 1$ , then  $G' = G + a_1b_1 + a_2b_2 + a_3b_3$  contains a  $K_6$ -minor.*

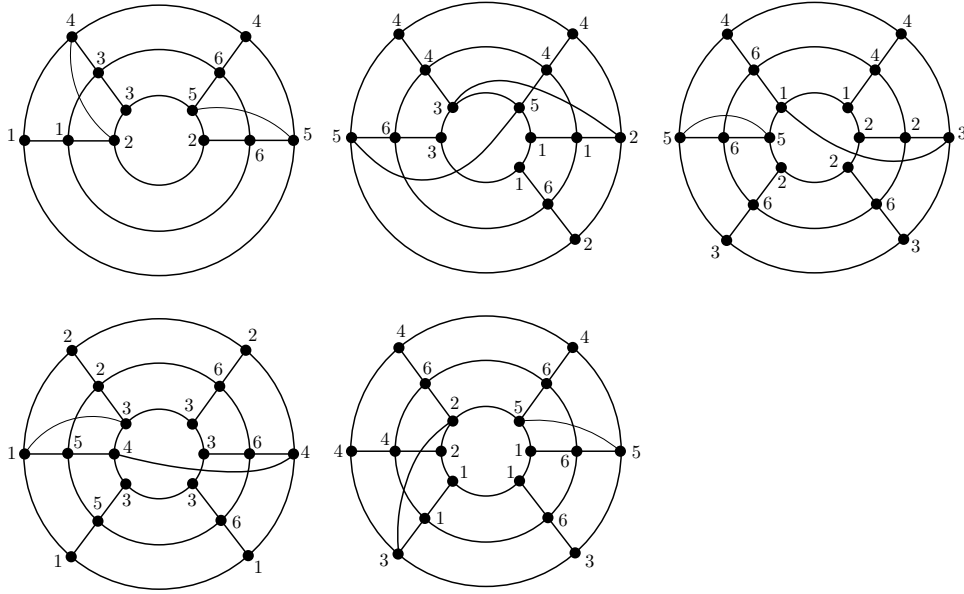


Figure 1: The graphs  $\Delta_1, \dots, \Delta_5$

*Proof.* Throughout the proof all indices are taken modulo  $k$ . We introduce the graphs  $\Delta_1, \dots, \Delta_5$  which are depicted in Figure 1. Each  $\Delta_i$  contains a  $K_6$ -minor as is evident by the labelling of the vertices in the figure. Thus, it suffices to prove that one of these graphs is a minor of  $G'$ .

By relabelling the paths  $P_i$  ( $0 \leq i < k$ ), we may assume that the vertices  $s_0, s_1, \dots, s_{k-1}$  (resp.,  $t_0, t_1, \dots, t_{k-1}$ ) appear on  $C_1$  (resp., on  $C_3$ ) in this cyclic order, and we consider the corresponding orientation of  $C_1$  and  $C_3$  as *clockwise* orientation. For  $i, j \in \{0, \dots, k-1\}$ , let  $C_1[s_i, s_j]$  (resp.,  $C_3[t_i, t_j]$ ) be the  $(s_i, s_j)$ -path (resp.,  $(t_i, t_j)$ -path) on  $C_1$  (resp.,  $C_3$ ) in the clockwise direction on the cycle. Set  $\mathcal{C} := C_1 \cup C_2 \cup C_3 \subseteq G$ .

By replacing  $G$  with a minor of  $G$ , we may assume that  $G$  is the union of the cycles  $C_1, C_2, C_3$  and the paths  $P_0, \dots, P_{k-1}$ . Moreover, any contraction of an edge on  $C_1$  or  $C_3$  either identifies two vertices in  $S \cup T$ , violates one of distance assumptions on  $a_1, a_2, b_1, b_2, b_3$ , or identifies  $b_1$  and  $b_2$ .

As a consequence, by (possibly) relabelling the paths, we may assume that the following conditions are satisfied:

- (1)  $a_1 = s_0, a_2 = s_j$  ( $3 \leq j \leq k-3$ ).
- (2) There are indices  $0 \leq \ell < r \leq k-1$  such that  $b_1 = t_\ell$  and  $b_2 = t_r$  or  $b_1 = t_r$  and  $b_2 = t_\ell$ .
- (3) If (ii) holds, then  $a_3 \in S$  and  $b_3 \in T$ .

*Proof of (i).* We proceed according to three cases.

**Case 1.** Suppose that  $\ell, r \in \{1, \dots, j-1\}$  or  $\ell, r \in \{j+1, \dots, k-1\}$ . By symmetry (i.e. after possibly changing the labelling of the paths to the reverse cyclic labelling), we may assume the former. Since  $r - \ell > 1$ ,  $s_{\ell+1} \neq s_r$ .

If  $b_1 = t_\ell$  and  $b_2 = t_r$ , then  $\Delta_1 \leq_m \mathcal{C} \cup P_{j+1} \cup P_{\ell+1} \cup P_r \cup P_j \cup \{a_1 b_1, a_2 b_2\}$ . To see this, consider the outer cycle of  $\Delta_1$  to correspond to  $C_1$  and the inner-most cycle to correspond to  $C_3$ . The four paths shown correspond in clockwise order, starting on the left,<sup>1</sup> to  $P_{j+1}, P_{\ell+1}, P_r$  and  $P_j$ , and the two crossed edge are obtained by contracting  $C_1(a_1, s_{\ell+1})$  and  $C_3(t_{j+1}, b_1)$ .

If  $b_1 = t_r$  and  $b_2 = t_\ell$  then  $\Delta_2 \leq_m \mathcal{C} \cup P_0 \cup P_\ell \cup P_r \cup P_j \cup P_{j+1} \cup \{a_1 b_1, a_2 b_2\}$ .

**Case 2.** Suppose  $\{r, \ell\} \cap \{0, j\} \neq \emptyset$ . By symmetry we may assume that  $\ell = 0$  and  $2 \leq r \leq j$ .

Suppose first that  $r < j$ . If  $b_1 = t_0$  and  $b_2 = t_r$ , then  $\Delta_1 \leq_m \mathcal{C} \cup P_{j+1} \cup P_1 \cup P_r \cup P_j \cup \{a_1 b_1, a_2 b_2\}$  (this is obtained after contracting  $C_1(a_1, s_1)$  and  $C_3(t_{j+1}, b_1)$ ). If  $b_1 = t_r$  and  $b_2 = t_0$ , then  $\Delta_2 \leq_m \mathcal{C} \cup P_{k-1} \cup P_1 \cup P_r \cup P_j \cup P_{j+1} \cup \{a_1 b_1, a_2 b_2\}$  (after contracting  $C_1(s_{k-1}, a_1)$  and  $C_3(b_2, t_1)$ ).

Suppose now that  $r = j$ . Since  $k \geq 7$ , we may assume by symmetry that  $j \leq k-4$ . If  $b_1 = t_0$  and  $b_2 = t_j$  then  $\Delta_1 \leq_m \mathcal{C} \cup P_{k-1} \cup P_1 \cup P_{j-1} \cup P_{j+1} \cup \{a_1 b_1, a_2 b_2\}$ . If  $b_1 = t_j$  and  $b_2 = t_0$ , then  $\Delta_2 \leq_m \mathcal{C} \cup P_{k-1} \cup P_1 \cup P_{j-1} \cup P_{j+1} \cup P_{j+2} \cup \{a_1 b_1, a_2 b_2\}$  (we contract  $C_1(s_{k-1}, a_1)$ ,  $C_1(a_2, s_{j+1})$ ,  $C_3(b_2, s_1)$  and  $C_3(t_{j-1}, b_1)$ ).

**Case 3.** Suppose that  $1 \leq \ell \leq j-1$  and  $j+1 \leq r \leq k-1$ . Then  $\Delta_1 \leq_m \mathcal{C} \cup P_r \cup P_0 \cup P_\ell \cup P_j \cup \{a_1 b_1, a_2 b_2\}$ .

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<sup>1</sup>We will stick with similar assumptions in the remaining cases:  $C_1$  and  $C_3$  will correspond to the outer and inner cycle, respectively, and the order of the paths in  $\Delta_i$  will correspond to the listed order, starting on the left and continuing clockwise.

*Proof of (ii).* We have  $r = \ell + 1$  (since by assumption  $\text{dist}_{(C_3, T)}(b_1, b_2) = 0$  and  $b_1 \neq b_2$ ). By symmetry, we may assume that  $0 \leq \ell \leq j - 1$ .

**Case 1.** Suppose that  $\ell = 0$  or  $\ell = j - 1$ . By symmetry we may assume that  $\ell = 0$  and then  $r = 1$ . If  $b_1 = t_0$  and  $b_2 = t_1$  then  $\Delta_3 \leq_m \mathcal{C} \cup P_0 \cup P_1 \cup P_{j-1} \cup P_j \cup P_{j+1} \cup P_{k-1} \cup \{a_1 b_1, a_2 b_2\}$ . If  $b_1 = t_1$  and  $b_2 = t_0$  then  $\Delta_4 \leq_m \mathcal{C} \cup P_0 \cup P_1 \cup P_{j-1} \cup P_j \cup P_{j+1} \cup P_{k-1} \cup \{a_1 b_1, a_2 b_2\}$ .

**Case 2.** Suppose that  $1 \leq \ell \leq j - 2$ . If  $b_1 = t_{\ell+1}$  and  $b_2 = t_\ell$ , then  $\Delta_2 \leq_m \mathcal{C} \cup P_0 \cup P_\ell \cup P_{\ell+1} \cup P_j \cup P_{j+1} \cup \{a_1 b_1, a_2 b_2\}$ .

Suppose now that  $b_1 = t_\ell$  and  $b_2 = t_{\ell+1}$ . We may assume that  $\ell = 1$  and  $\ell + 2 = j$ . For if  $\ell \neq 1$ , then  $\Delta_5 \leq_m \mathcal{C} \cup P_1 \cup P_\ell \cup P_{\ell+1} \cup P_j \cup P_{j+1} \cup P_0 \cup \{a_1 b_1, a_2 b_2\}$ , and the case when  $\ell \neq j - 2$  is symmetric to the case when  $\ell \neq 1$ .

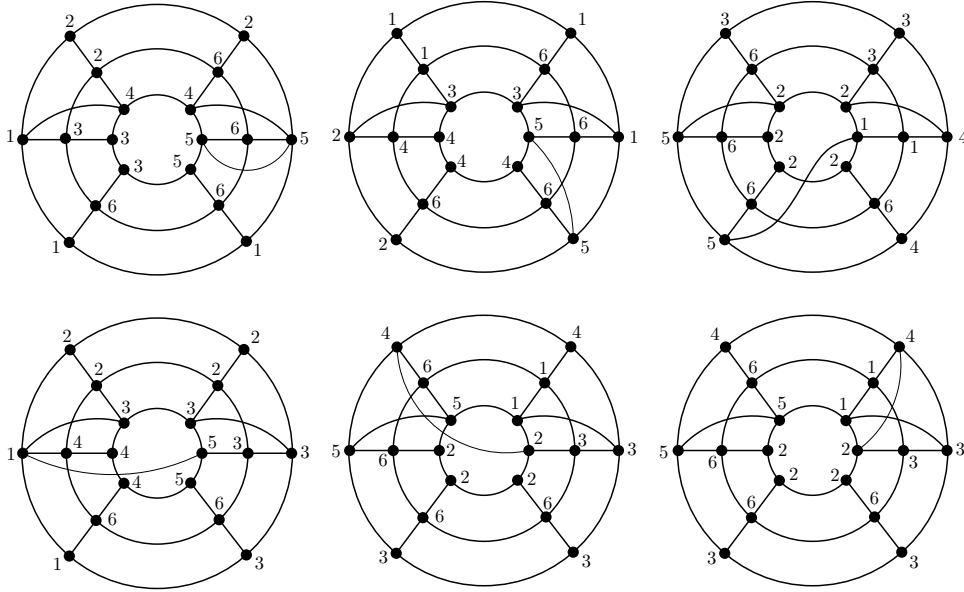


Figure 2: The graphs obtained in Case 2 of proof of (ii) contain  $K_6$ -minors

Hence, we are left with the case where  $\ell = 1$  and  $j = 3$ . By assumption,  $b_3 \in T \setminus \{t_1, t_2\}$ . Suppose that  $b_3 \in \{t_5, \dots, t_{k-2}\}$ . Then  $\text{dist}_{(C_3, T)}(b_3, b_1) \geq 2$  and  $\text{dist}_{(C_3, T)}(b_3, b_2) \geq 2$ . In addition, since  $\text{dist}_{(C_1, S)}(a_1, a_2) \geq 2$ , there is  $z \in \{1, 2\}$  such that  $\text{dist}_{(C_1, S)}(a_3, a_z) \geq 1$ . Then the proof follows by the proof of (i) by interchanging the roles of  $C_1$  and  $C_3$  and  $S$  and  $T$ , with  $b_3$  and  $b_z$  playing the role of  $a_1$  and  $a_2$ , and  $a_3$  and  $a_z$  playing the role of  $b_1$  and  $b_2$ , respectively. Thus, we may assume that  $b_3 \in \{t_0, t_3, t_4, t_{k-1}\}$ .

Suppose that  $b_3 \in \{t_0, t_3\}$ . By symmetry, we may assume that  $b_3 = t_3$ . Let  $H := \mathcal{C} \cup \{P_0, P_1, P_2, P_3, P_4, P_5\} \cup \{a_1 b_1, a_2 b_2, a_3 b_3\}$ . By contracting edges on  $C_1$ , we obtain a minor  $H'$  of  $H$  such that  $a_3 \in \{s_0, s_1, \dots, s_5\}$ . For each of these six possibilities for  $a_3$ , we see that  $H'$  contains a  $K_6$ -minor (see Figure 2).

Finally, suppose that  $b_3 \in \{t_4, t_{k-1}\}$ . By symmetry, we may assume that  $b_3 = t_4$ . Let  $H := \mathcal{C} \cup \{P_0, P_1, P_2, P_3, P_{k-2}, P_{k-1}\} \cup \{a_1 b_1, a_2 b_2, a_3 b_3\}$ . Let  $H'$  be obtained from  $H$  by contracting  $C_3(t_3, b_3)$ . The proof now follows as in the previous paragraph.  $\square$

## 5 The projective plane

In this section we present the proof of Theorem 1.1.

The projective plane contains graphs of face-width 3 that do not contain  $K_6$  as a minor. In fact the graphs obtained from  $K_6$  by performing one or more  $\Delta Y$ -transformations<sup>2</sup> on facial triangles of  $K_6$  provide such examples. On the other hand, face-width four forces  $K_6$  minor as claimed by Theorem 1.1. In this section we give a proof of this theorem.

It suffices to prove Theorem 1.1 for minor-minimal graphs embedded in the projective plane with face-width 4. It was proved by Randby [14] that every such graph can be obtained from the projective  $4 \times 4$  grid (the first graph depicted in Figure 3) by a series of  $Y\Delta$  and  $\Delta Y$ -transformations. Let  $\mathcal{G}_4$  be the family of such graphs. It is known [4] that  $\mathcal{G}_4$  contains precisely 270 graphs.

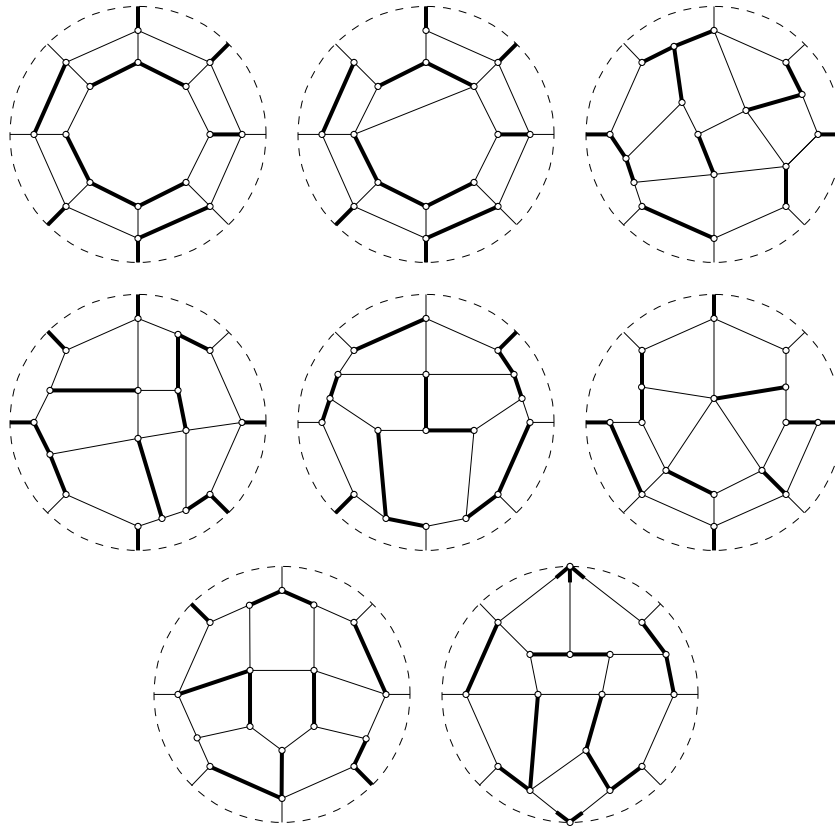


Figure 3: The triangle-free graphs in  $\mathcal{G}_4$  contain  $K_6$  minors

It is easy to see that if  $G$  is obtained from  $H$  by a  $\Delta Y$ -transformation and  $G$  has a  $K_6$  minor, then so does  $H$ . Therefore it suffices to prove that all triangle-free graphs in  $\mathcal{G}_4$

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<sup>2</sup>We say that a graph  $H$  is obtained from a graph  $G$  by a  $\Delta Y$ -transformation if the edges of a triangle  $T = uvw$  are removed from  $G$  and replaced by a new vertex  $y$  and three edges joining  $y$  with each of  $u, v, w$ . The inverse operation is said to be a  $Y\Delta$ -transformation.

contain  $K_6$  as a minor. To justify this conclusion, note that graphs in  $\mathcal{G}_4$  have face-width 4; thus every triangle in such a graph  $G$  is facial in any embedding of  $G$  on the projective plane. Also observe that every  $\Delta Y$ -transformation increases the number of vertices, thus performing these transformations as long as possible, we end up with a triangle-free graph.

Examining the 270 graphs in  $\mathcal{G}_4$ , we found that precisely eight of them are triangle-free. They are shown in Figure 3 (drawn in the projective plane), where also a  $K_6$  minor is exhibited for each of them (the thick edges should be contracted in order to obtain a  $K_6$  minor). This observation completes the proof of Theorem 1.1.

## 6 Proof of the main result

In this section we prove Theorem 1.2. Let  $G$  and  $\Sigma$  be as in the theorem. By Theorem 2.7, we may assume that  $G$  is 3-connected and that  $\text{fw}(G) \geq 6$ . Let  $\Lambda$  and  $n$  be as in (H1). By Theorem 2.4,  $\Lambda$  is clean. We shall also assume that  $x_i \in V(G)$  ( $i = 0, \dots, n-1$ ), for if  $x_i$  is an edge then we contract  $x_i$  and work with the resulting minor of  $G$ . The only danger is that such contractions lower  $\text{nsfw}(G)$ . However, this will not be a problem, since any further arguments involving large  $\text{nsfw}(G)$  will refer to the original graph.

Let  $\Sigma$  be a cylinder with cuffs  $F_1$  and  $F_2$ . Let  $G$  be a graph embedded on  $\Sigma$ , and let  $n \geq 2$  be an integer. Let  $C_1, \dots, C_n$  be pairwise disjoint homotopic cycles in  $G$  such that  $F_1 = C_1$ ,  $F_2 = C_n$  and  $C_1, \dots, C_n$  appear along  $\Sigma$  in order. For  $i = 1, \dots, n-1$ , we say that  $C_{i+1}$  is *tight* in  $G$  with respect to  $C_i$ , if there does not exist a  $(C_{i+1})$ -path  $P$  that is disjoint from  $C_{i+1}$  except for its two ends,  $P$  is disjoint from  $C_i$ , and  $P$  is embedded in the sub-cylinder of  $\Sigma$  bounded by  $C_i$  and  $C_{i+1}$ .

The proof proceeds according two two cases, depending whether  $\Lambda$  is 2-sided or 1-sided.

### 6.1 Proof of Theorem 1.2 when $\Lambda$ is 2-sided

As  $\Gamma(\Lambda)$  is 2-sided, then as we traverse along  $\Gamma(\Lambda)$  on  $\Sigma$ , one side is naturally the “left-hand side” and the other is the “right-hand side”. The curve  $\Gamma(\Lambda)$  splits each face  $F_i$  into two closed disks. Each of these closed disks is bounded by the portion of  $\Gamma(\Lambda)$  in  $F_i$  and a part of the boundary of  $F_i$ . For  $i = 0, \dots, n-1$ , let  $\partial_L(F_i)$  ( $\partial_R(F_i)$ ) be the portion of the boundary of  $F_i$  to the left (right) of  $\Gamma(\Lambda)$ . Then each of  $\partial_L(F_i)$  and  $\partial_R(F_i)$  is a path in  $G$  from  $x_i$  to  $x_{i+1}$  (indices modulo  $n$ ). All these paths are pairwise disjoint except for their ends. Set  $C_L(\Lambda) := \bigcup_{i=0}^{n-1} \partial_L(F_i)$  and  $C_R(\Lambda) := \bigcup_{i=0}^{n-1} \partial_R(F_i)$ . Since  $\Lambda$  is clean, each of  $C_L(\Lambda)$  and  $C_R(\Lambda)$  is a cycle in  $G$ .

Cutting  $\Sigma$  along  $\Gamma(\Lambda)$ , results in a new graph  $G'$  embedded on  $\Sigma'$ , where  $\Sigma'$  is the surface obtained from  $\Sigma$  by cutting along  $\Gamma(\Lambda)$  and capping off the resulting two cuffs. Let  $F_L$  and  $F_R$  be the two added faces of  $G'$  whose boundaries coincide with  $C_L(\Lambda)$  and  $C_R(\Lambda)$ , respectively. By Theorem 2.5, we have  $\text{nsfw}(G') \geq 4$  and  $\text{fw}(G') \geq 3$ , since  $\text{nsfw}(G) \geq 7$  and  $\text{fw}(G) \geq 6$ . Let us now apply Theorem 2.6 to  $G'$  and its faces  $F_L$  and  $F_R$ , respectively. Let  $B_1(F_L)$ ,  $C_1(F_L)$ ,  $B_1(F_R)$  and  $C_1(F_R)$  be the disks (cycles) as obtained by the application of Theorem 2.6 and the facts that  $\text{nsfw}(G') \geq 4$  and  $\text{fw}(G') \geq 3$ . Set  $\Omega_R := C_1(F_R)$  and

$\Omega_L := C_1(F_L)$ . Note that  $\Omega_L$  and  $\Omega_R$  are homotopic to  $\Gamma(\Lambda)$  and that they bound a cylinder containing  $\Lambda$ . By possibly altering  $\Omega_R$  and  $\Omega_L$ , we may assume that the following holds:

**Lemma 6.1.** *In  $G'$ , the cycle  $\Omega_L$  (resp.,  $\Omega_R$ ) is tight with respect to  $F_L$  (resp.,  $F_R$ ), and  $\Omega_L \subseteq B_1(F_L)$  (resp.,  $\Omega_R \subseteq B_1(F_R)$ ).*

Next we observe that

**Lemma 6.2.**  *$\Omega_R$  and  $\Omega_L$  are disjoint.*

*Proof.* For suppose not, then we let  $v \in V(\Omega_R \cap \Omega_L)$ . By the definition of  $B_1(F_R)$  and  $B_1(F_L)$ ,  $v$  is co-facial with some vertex of  $F_R$ , say  $w_R$ , and some vertex of  $F_L$ , say  $w_L$ . In  $G$ , the vertices  $v, w_R, w_L$ , define a face-chain  $\Lambda'$  of length two (not closed), starting and ending in  $\Lambda$ , whose faces are distinct from the faces of  $\Lambda$ . Let  $\Lambda_1$  and  $\Lambda_2$  be the two face-subchains<sup>3</sup> in  $\Lambda$  with ends  $w_L$  and  $w_R$ . As  $\Gamma(\Lambda')$  connects the left side of  $\Gamma(\Lambda)$  with its right side, we see that both  $\Gamma(\Lambda' \cup \Lambda_1)$  and  $\Gamma(\Lambda' \cup \Lambda_2)$  are surface non-separating in  $G$ .

To obtain a contradiction, note that by Theorem 2.2 (with  $k = 2$ ), one of  $\Lambda' \cup \Lambda_1$  and  $\Lambda' \cup \Lambda_2$  is of length at most 6. Since both  $\Gamma(\Lambda' \cup \Lambda_1)$  and  $\Gamma(\Lambda' \cup \Lambda_2)$  are surface non-separating in  $G$ , we have a contradiction to the assumption that  $\text{nsfw}(G) \geq 7$ .  $\square$

In  $G$ , the cycles  $\Omega_R$  and  $\Omega_L$  are homotopic to  $\Gamma(\Lambda)$  (and homotopic to each other). Therefore, there exists  $\Sigma' \subseteq \Sigma$  such that  $\Sigma'$  is homeomorphic to a cylinder, the cuffs of which coincide with  $\Omega_R$  and  $\Omega_L$ , and  $\Gamma(\Lambda) \subseteq \Sigma'$ . Let  $G(\Omega_L, \Omega_R) \subseteq G$ , be the subgraph of  $G$  embedded in  $\Sigma'$  (including  $\Omega_L$  and  $\Omega_R$ ).

Let  $\mathcal{Q} = \{Q_1, Q_2, \dots\}$  be a set of pairwise disjoint paths, such that each  $Q_i$  is an  $(\Omega_L, \Omega_R)$ -path, disjoint from  $G(\Omega_L, \Omega_R)$  except for its ends. If  $\mathcal{Q}$  is of maximum cardinality, then we say that  $\mathcal{Q}$  is an *exterior*  $(\Omega_L, \Omega_R)$ -linkage. By Lemma 3.1,  $|\mathcal{Q}| \geq \text{nsfw}(G) \geq 7$ .

By two applications of Lemma 3.4 and using Lemma 6.1, we see that  $G(\Omega_L, \Omega_R)$  contains a set  $P_0^R, \dots, P_{n-1}^R$  and  $P_0^L, \dots, P_{n-1}^L$  of pairwise disjoint paths, satisfying the following properties:

- (1) For  $i = 0, \dots, n-1$ ,  $P_i^L$  (resp.,  $P_i^R$ ) has ends  $x_i$  and  $l_i \in \Omega_L$  (resp.,  $r_i \in \Omega_R$ ) and is otherwise disjoint from  $\Omega_L$  ( $\Omega_R$ ) and  $X(\Lambda)$ . For  $i = 0, \dots, n-1$ , set  $P_i := P_i^L \cup P_i^R$ . Note that  $P_i$  is an  $(l_i, r_i)$ -path contained in  $G(\Omega_L, \Omega_R)$ . Also note that the vertices  $r_0, r_1, \dots, r_{n-1}$  ( $l_0, l_1, \dots, l_{n-1}$ ) appear on  $\Omega_R$  ( $\Omega_L$ ) in order.
- (2) Let  $X \in \{L, R\}$ . For  $i = 0, \dots, n-1$ ,  $P_i$  is disjoint from  $\partial_X(F_j)$ , if  $j \neq \{i-1, i\}$  (indices modulo  $n$ ). In addition, we may assume that  $P_i - x_i$  intersects at most one of  $\partial_X(F_i)$  and  $\partial_X(F_{i-1})$ . If  $P_i - x_i$  intersects  $F_i$  we say that  $F_i$  is the *X-support* of  $P_i$ , otherwise  $F_{i-1}$  is the *X-support* of  $P_i$ .

A set  $\mathcal{P} = \{P_0, \dots, P_{n-1}\}$  of paths satisfying properties (1) and (2) above, is called an *internal*  $(\Omega_L, \Omega_R)$ -linkage. Figure 4 shows part of an internal linkage and the corresponding notation as used in the sequel.

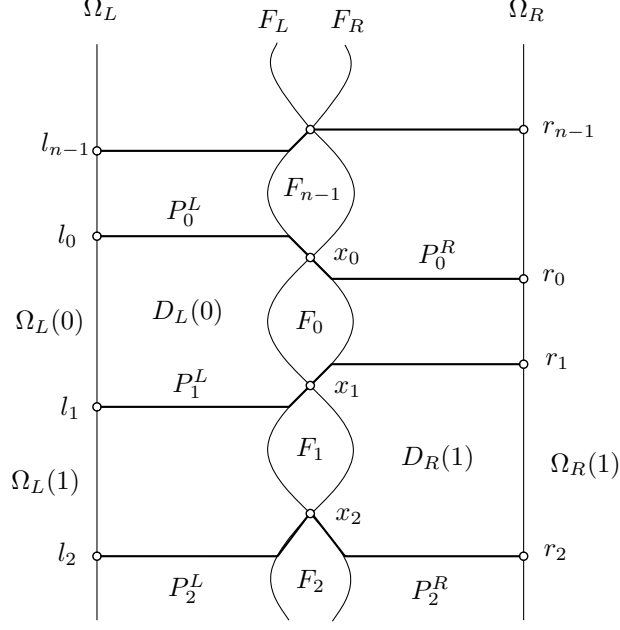


Figure 4: Internal  $(\Omega_L, \Omega_R)$ -linkage

For  $i = 0, \dots, n-1$ , let  $\Omega_R(i)$  (resp.,  $\Omega_L(i)$ ) be the path on  $\Omega_R$  (resp.,  $\Omega_L$ ) from  $r_i$  to  $r_{i+1}$  (resp.,  $l_i$  to  $l_{i+1}$ ) not passing thorough  $r_{i+2}$  (resp.,  $l_{i+2}$ ).

Let  $X \in \{R, L\}$ . For a subset of indices  $I = \{i_0, \dots, i_{|I|-1}\} \subseteq \{0, \dots, n-1\}$ , let  $\mathcal{F}_X(\mathcal{P}, I) \subseteq F(\Lambda)$  be a set of consecutive faces of  $\Lambda$  satisfying the following:

- (1) For every vertex  $v \in \cup_{i \in I} \Omega_X(i)$  there exists  $f \in \mathcal{F}_X(\mathcal{P}, I)$  such that  $v$  is co-facial with some vertex of  $f$ .
- (2) Subject to (1),  $|\mathcal{F}_X(\mathcal{P}, I)|$  is minimum.

Observe that  $\mathcal{F}_X(\mathcal{P}, I)$  exists by Lemma 6.1. Of special interest is the case when  $I = \{i\}$  or  $I = \{i, i+1\}$  for some  $0 \leq i \leq n-1$ . Note that  $1 \leq |\mathcal{F}_X(\mathcal{P}, \{i\})| \leq 3$  and  $1 \leq |\mathcal{F}_X(\mathcal{P}, \{i, i+1\})| \leq 4$ .

For  $i = 0, \dots, n-1$  and  $X \in \{R, L\}$ , let  $D_X(i)$  be the closed disk bounded by  $\Omega_X(i)$ ,  $P_i^X$ ,  $P_{i+1}^X$  and a path in  $C_X(\Lambda)$  on the boundary of the faces in  $\mathcal{F}_X(\mathcal{P}, \{i\})$ .

The following is a direct consequence of the definition of  $\mathcal{F}_X(\mathcal{P}, \{i\})$ .

**Lemma 6.3.** *For  $i = 0, \dots, n-1$ , each of  $r_i$  and  $r_{i+1}$  (resp.,  $l_i$  and  $l_{i+1}$ ) is co-facial in  $G(\Omega_L, \Omega_R)$  with some vertex in  $V(\mathcal{F}_R(\mathcal{P}, \{i\}))$  (resp.,  $V(\mathcal{F}_L(\mathcal{P}, \{i\}))$ ).*

A *system* is a pair  $(\mathcal{Q}, \mathcal{P})$ , where  $\mathcal{Q}$  is an exterior  $(\Omega_L, \Omega_R)$ -linkage and  $\mathcal{P}$  is an interior  $(\Omega_L, \Omega_R)$ -linkage. For  $X \in \{L, R\}$  and a subset  $\mathcal{A} \subseteq \mathcal{Q} \cup \mathcal{P}$  of paths, we denote by

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<sup>3</sup>Strictly speaking,  $\Lambda_1$  and  $\Lambda_2$  need not be subchains since  $w_L$  and  $w_R$  need not be constituents of  $\Lambda$ . But all other faces and vertices in  $\Lambda_1$  and  $\Lambda_2$  are taken from  $\Lambda$ .

$Ends(\mathcal{A}, \Omega_X)$  the set of endvertices of the paths in  $\mathcal{A}$  contained in  $\Omega_X$ . If  $A$  is a single path, we set  $End(A, \Omega_X) = Ends(\{A\}, \Omega_X)$ . The following is the key ingredient in the proof.

**Lemma 6.4.** *Let  $k \geq n \geq 7$ , and let  $\mathfrak{S} = (\mathcal{Q}, \mathcal{P})$  be a system, where  $\mathcal{Q} = \{Q_0, \dots, Q_{k-1}\}$  and  $\mathcal{P} = \{P_0, \dots, P_{n-1}\}$ . Let  $Z \in \{L, R\}$  and  $Y \in \{L, R\} \setminus Z$ . Then there exists a system  $(\mathcal{Q}', \mathcal{P}')$ , where  $\mathcal{Q}' = \{Q'_0, \dots, Q'_{k-1}\}$  and  $\mathcal{P}' = \{P'_0, \dots, P'_{n-1}\}$  satisfying the following:*

- (P1) *For  $i = 0, \dots, n-1$ ,  $P'_i{}^Z = P_i^Z$ .*
- (P2) *For  $i = 0, \dots, k-1$ ,  $End(Q_i, \Omega_Z) = End(Q'_i, \Omega_Z)$ .*
- (P3) *There exist paths  $A, B \in \mathcal{Q}'$ , such that  $End(A, \Omega_Y)$  and  $End(B, \Omega_Y)$  are at distance at least two on  $\Omega_Y$  with respect to  $Ends(\mathcal{P}', \Omega_Y)$ .*

*Proof.* Assume for a contradiction that the claim is false. By symmetry we may assume that  $Z = R$  and  $Y = L$ .

For every system  $(\mathcal{Q}', \mathcal{P}')$  satisfying (P1) and (P2), let  $S \subseteq \Omega_L$  be a minimal segment on the cycle  $\Omega_L$  such that  $Ends(\mathcal{Q}', \Omega_L) \subseteq V(S)$  and  $S$  contains in its interior at most one vertex from the set  $Ends(\mathcal{P}', \Omega_L)$ . Note that  $S$  exists since otherwise (P3) would hold. Let  $T \subseteq \Omega_L$  be the minimal segment on  $\Omega_L$  containing  $S$  such that the endpoints of  $T$  are in  $Ends(\mathcal{P}', \Omega_L)$ . Among all systems  $(\mathcal{Q}', \mathcal{P}')$ , we choose one with the following properties:

- (J0)  $(\mathcal{Q}', \mathcal{P}')$  satisfies (P1) and (P2).
- (J1) The number of vertices of  $Ends(\mathcal{P}', \Omega_L)$  contained in  $T$  is as large as possible and  $|V(T) \setminus V(S)|$  is minimum.

Such a choice of  $(\mathcal{Q}', \mathcal{P}')$  is clearly possible as  $(\mathcal{Q}, \mathcal{P})$  satisfies (P1) and (P2). By symmetry, we may assume that  $T = \Omega_L(0)$  or  $T = \Omega_L(0) \cup \Omega_L(1)$  and that  $|Ends(\mathcal{Q}, \Omega_L) \cap V(\Omega_L(0))| \geq 4$ . To simplify notation, we may also assume that  $(\mathcal{Q}, \mathcal{P}) = (\mathcal{Q}', \mathcal{P}')$ .

Set  $X := \Omega_L(n-1) \cup \Omega_L(0) \cup \Omega_L(1)$ . Then  $X \subseteq \Omega_L$  is an  $(l_{n-1}, l_2)$ -path on  $\Omega_L$  containing  $l_0$ . Let  $\bar{X}$  be the other  $(l_{n-1}, l_2)$ -path on  $\Omega_L$  such that  $X \cup \bar{X} = \Omega_L$  and  $X$  and  $\bar{X}$  are disjoint except for their ends. We may further assume that  $\mathfrak{S}'$  is chosen so that

- (J2) Subject to (J1),  $|V(X)|$  is minimum.

Observe that by the maximality of the number of disjoint paths in  $\mathcal{Q}$ , there does not exist an  $(int(\bar{X}), \Omega_R)$ -path which is internally disjoint from  $G(\Omega_L, \Omega_R)$ . Let us apply Theorem 3.3 to  $G$  with  $\mathcal{Q}$ ,  $X$  and its ends (that is, the endpoints of  $X$  playing the role of  $w$  and  $w'$  in Theorem 3.3). Note that outcome (b) of Theorem 3.3 is obtained. Let  $u, v \in V(X)$  and  $f$  be as promised to exist by Theorem 3.3(b). Without loss of generality, assume that  $v$  is closer to  $l_{n-1}$  than  $u$  on  $X$ .

The property of  $f$  stated in Theorem 3.3 implies the following.

- (1) Every face-chain in  $G(\Omega_L, \Omega_R)$  with ends  $v$  and  $u$  has length at least 6.



To see this, suppose that  $\Lambda'$  is a face-chain in  $G(\Omega_L, \Omega_R)$  of length at most five with ends  $u$  and  $v$ . Then  $\Lambda' \cup f$  is a closed face-chain in  $G$  of length at most six, and by the property of  $f$ ,  $\Gamma(\Lambda' \cup f)$  is surface non-separating; contradicting the assumption that  $\text{nsfw}(G) \geq 7$ . This proves (1).

Property (1) immediately implies that

$$(2) \quad v \in V(\Omega_L(n-1)) \text{ and } u \in V(\Omega_L(1)).$$

Next we claim the following:

- (3) In  $G(\Omega_L, \Omega_R)$  there exist face chains  $g_v, g_u$  of length at most two such that  $g_v$  has ends  $v, l_0$ , and  $g_u$  has ends  $l_1, u$ . Moreover, if  $g_v$  (resp.,  $g_u$ ) is of length two, then  $v$  (resp.,  $u$ ) is co-facial with some vertex in the  $L$ -support of  $P_0$  (resp.,  $P_1$ ).

We will prove existence of  $g_u$  (the proof for  $g_v$  is exactly the same; in fact it is even easier since no  $Q \in \mathcal{Q}$  has an end in the interior of  $\Omega_L(n-1)$ ).

Let  $j \in \{0, 1\}$  so that  $F_j$  is the  $L$ -support of  $P_1$ . Let  $w$  be the end of the path  $P_1 \cap \partial_L(F_j)$  of  $P_1$  so that  $w \neq x_1$ , unless the path  $P_1 \cap \partial_L(F_j)$  is the single vertex  $x_1$ . It suffices to show that every vertex  $x \in \Omega_L(l_1, u)$  is of degree two in  $G(\Omega_R, \Omega_L)$  or has no neighbors in  $G(\Omega_R, \Omega_L)$  except for  $w$  and the two neighbors of  $x$  on the cycle  $\Omega_L$ . (Note that  $x$  is adjacent in  $G(\Omega_L, \Omega_R)$  with only two vertices in  $\Omega_L$ , since  $\Omega_L$  is tight.)

Suppose to the contrary that there exists  $x \in \Omega_L(l_1, u)$  such that  $x$  has a neighbor in  $G(\Omega_L, \Omega_R)$  that is distinct from  $w$  and from the two neighbors of  $x$  on the cycle  $\Omega_L$ . Since  $G$  is 3-connected (and hence  $G - w$  is 2-connected), by Menger's theorem there exists a  $(\partial D_L(1))$ -path  $P$  with ends  $x$  and  $y$ , where  $y \in V(\partial D_L(1)) \setminus \{w\}$ . Since  $\Omega_L$  is tight, we have  $y \in V(F_L)$ .

**Case 1.** Suppose  $j = 1$ . By Lemma 6.1,  $y \in V(F_1 \cup F_2) \setminus \{w\}$ . Let  $P'_2$  be the path obtained from  $P_2$  by rerouting  $P_2^L$  so that it passes via  $P$ . Let  $\mathcal{P}'$  be the new collection of paths. We claim that  $\mathcal{P}'$  contradicts our choice of  $\{\mathcal{Q}, \mathcal{P}\}$ .

If  $\text{Ends}(\mathcal{Q}, \Omega_L(l_1, l_2)) = \emptyset$ , then  $\{\mathcal{Q}, \mathcal{P}'\}$  contradicts (J2). Suppose that  $\text{Ends}(\mathcal{Q}, \Omega_L(x, u)) \neq \emptyset$  for some  $Q \in \mathcal{Q}$ , and let  $Q'$  be a path in  $\mathcal{Q}$  with an end in  $\Omega_L(l_0, l_1)$ . Such a path exists since  $|\text{Ends}(\mathcal{Q}, \Omega_L(0))| \geq 3$  and hence  $|\text{Ends}(\mathcal{Q}, \Omega_L[l_0, l_1])| \geq 1$ . Then  $Q$  and  $Q'$  are at distance at least two with respect to  $\text{Ends}(\mathcal{P}', \Omega_L)$  and hence  $\{\mathcal{Q}, \mathcal{P}'\}$  satisfies (P3). It follows that  $\text{Ends}(\mathcal{Q}, \Omega_L) \subseteq \Omega_L(0) \cup \Omega_L[l_1, x]$ , but then  $\{\mathcal{Q}, \mathcal{P}'\}$  contradicts (J1).

**Case 2.**  $j = 0$ . By Lemma 6.1,  $y \in V(F_0 \cup F_1 \cup F_2) \setminus \{w\}$ . We first observe that  $u$  is not co-facial in  $G(\Omega_L, \Omega_R)$  with any vertex in  $V(F_0)$ . For suppose  $u$  is co-facial in  $G(\Omega_L, \Omega_R)$  with  $F_0$ , say via a face  $g_0$ . By Lemma 6.3,  $v$  is co-facial in  $G(\Omega_L, \Omega_R)$ , say via a face  $g_1$ , with some vertex of  $V(\partial_L(F_i))$ , for some  $i \in \{n-2, n-1, 0\}$ . Then using  $g_0, g_1, F_{n-2}, F_{n-1}$  and  $F_0$  we can construct a face-chain of length at most five in  $G(\Omega_L, \Omega_R)$  with ends  $u$  and  $v$ , contradicting (1).

If  $y \in V(F_1 \cup F_2) \setminus \{x_1\}$ , the proof proceeds exactly as in Case 1 (by replacing  $P_2^L$  by another path using  $P'$  and thus obtaining a contradiction). Hence, we may assume that  $y \in V(F_0) \setminus \{w\}$ . Since  $u$  is not co-facial with any vertex in  $V(F_0)$ , Lemma 6.1 implies that

there exists a path  $P'$  in  $D_L(1)$  with one end in  $V(F_1 \cup F_2) \setminus \{x_1\}$  and the other end in  $V((P - F_0) \cup \Omega_L[x, u])$ . We then see that there exists a path  $P''$  with one end in  $V(\Omega_L[x, u])$  and one end in  $V(F_1 \cup F_2) \setminus \{x_1\}$ , and the proof again proceeds exactly as in Case 1 with  $P''$  playing the role of  $P$ . This proves (3).

Observe that at least one of  $P_0$  and  $P_1$  is not  $L$ -supported by  $F_0$ . For if both  $P_0$  and  $P_1$  are  $L$ -supported by  $F_0$ , then by Lemma 6.3, each of  $l_0$  and  $l_1$  is co-facial with some vertex in  $V(F_0)$ . By (3), there is a face-chain of length at most two from  $v$  (resp.,  $u$ ) to a vertex in  $V(F_0)$ , since by (3),  $v$  is either co-facial with  $l_0$  (resp.,  $u$  is co-facial with  $l_1$ ) in  $G(\Omega_L, \Omega_R)$  or co-facial with a vertex in  $V(F_0)$ . Combining these faces together with  $F_0$ , we obtain a face-chain from  $v$  to  $u$  in  $G(\Omega_L, \Omega_R)$  that is of length at most 5, contradicting (1).

We will assume henceforth that  $P_1$  is  $L$ -supported by  $F_1$  (if  $P_1$  is  $L$ -supported by  $F_0$ , the proof follows the same arguments). Now we distinguish two cases: either  $P_0$  is  $L$ -supported by  $F_0$  or by  $F_{n-1}$ . We consider the latter case (the former case is proved by the same arguments).

By (1) and (3),  $l_0$  and  $l_1$  are not co-facial in  $G(\Omega_R, \Omega_L)$ . Hence there exists  $x \in V(\Omega_L(l_0, l_1))$  of degree at least three in  $G(\Omega_R, \Omega_L)$ . By Menger's theorem and since  $\Omega_L$  is tight, there is  $(\partial D_L(0))$ -path  $P$  in  $D_L(0)$  with ends  $x$  and  $y$ , where  $y \in \partial D_L(0) \cap (F_{n-1} \cup F_0 \cup F_1)$ .

**Case 1.** Suppose that  $y \in V(F_0) \setminus \{x_0, x_1\}$ . Let  $P'_0$  (resp.,  $P'_1$ ) be obtained from  $P_0$  (resp.,  $P_1$ ) by re-rerouting it so that it passes via  $P$  rather than via  $P_0^L$  (resp.,  $P_1^L$ ). Let  $\mathcal{P}' = (\mathcal{P} \setminus \{P_0\}) \cup \{P'_0\}$  and let  $\mathcal{P}'' = (\mathcal{P} \setminus \{P_1\}) \cup \{P'_1\}$ . We claim that one of  $(\mathcal{Q}, \mathcal{P}')$  or  $(\mathcal{Q}, \mathcal{P}'')$  contradicts our choice of  $(\mathcal{Q}, \mathcal{P})$ . We argue as follows.

We may assume that  $\text{Ends}(\mathcal{Q}, \Omega_L[l_0, x]) \neq \emptyset$ , for otherwise  $(\mathcal{Q}, \mathcal{P}')$  contradicts (J1). Further, we may assume that  $\text{Ends}(\mathcal{Q}, \Omega_L[l_1, l_2]) = \emptyset$ , for otherwise  $(\mathcal{Q}, \mathcal{P}')$  satisfies (P3) (since  $\text{Ends}(\mathcal{Q}, \Omega_L[l_0, x]) \neq \emptyset$ ). Now it is easy to see that  $(\mathcal{Q}, \mathcal{P}'')$  contradicts one of the two conditions stated in (J1).

**Case 2.** Suppose that  $y \in V(F_1 \cup F_{n-1})$ . We may assume that  $y \in V(F_1)$  (if  $y \in V(F_{n-1})$  the proof follows by the same arguments). Let  $w$  be the end of the path  $P_0 \cap \partial_L(F_{n-1})$  so that  $w \neq x_0$  unless  $V(P_0 \cap \partial_L(F_{n-1})) = \{x_0\}$ . We proceed according to two cases, depending on whether  $l_0$  is co-facial with a vertex in  $V(F_1)$  or not.

**Case 2.1.** If  $l_0$  is not co-facial with a vertex in  $V(F_1)$ , there exists a path  $R$  in  $D_L(0)$  with one end in  $V(\Omega_L(l_0, x])$  and the other in  $V(F_{n-1} \cup (F_0 - x_1))$  (since  $\Omega_L$  is tight). Let  $P'_0$  (resp.,  $P'_1$ ) be obtained from  $P_0$  (resp.,  $P_1$ ) by re-rerouting it so that it passes via  $R$  (resp.,  $P$ ) rather than via  $P_0^L$  (resp.,  $P_1^L$ ). The proof now proceed exactly as in Case (1).

**Case 2.2.** Suppose that  $l_0$  is co-facial with some vertex in  $V(F_1)$  via a face  $f_0$ . By (1) and (3),  $g_v$  must be a face-chain of length two. By (the proof of) (3),  $v$  and  $w$  are co-facial in  $G(\Omega_L, \Omega_R)$  and there exists a vertex  $z \in \Omega_L(v, l_0)$  such that  $zw \in E(G(\Omega_L, \Omega_R))$ . Hence by (1),  $w$  is not co-facial with a vertex in  $V(F_1)$ . Thus, there exists a path  $R$  in  $D_L(0)$  with one end in  $V(F_{n-1} \cup (F_0 - x_1))$  and the other end, say  $a$ , in  $V(P_0^L - V(F_{n-1}))$ . Note that the path  $R$  cannot end up in  $\Omega_L(0) \setminus \{l_0\}$  because of the face  $f_0$ .

Let  $P'_0$  (resp.,  $P'_{n-1}$ ) be obtained from  $P_0$  (resp.,  $P_{n-1}$ ) by re-rerouting it so that it passes via  $R$  (resp., the edge  $wz$ ) rather than via  $P_0^L$  (resp.,  $P_{n-1}^L$ ). Let  $\mathcal{P}'$  be the new collection of

paths obtained by replacing  $P_0$  and  $P_{n-1}$  with  $P'_0$  and  $P'_{n-1}$ . Then  $(\mathcal{Q}, \mathcal{P}')$  contradicts (J2). This completes the proof of Lemma 6.4.  $\square$

Now we can complete the proof of Theorem 1.2 when  $\Lambda$  is 2-sided. Let  $(\mathcal{Q}, \mathcal{P})$  be a system. By two applications of Lemma 6.4, we may assume that  $(\mathcal{Q}, \mathcal{P})$  satisfies (P3) for  $X \in \{L, R\}$ .

Let  $A, B \in \mathcal{Q}$ , such that  $\text{End}(A, \Omega_L)$  and  $\text{End}(B, \Omega_L)$  are at distance at least two with respect to  $\text{Ends}(\mathcal{P}, \Omega_L)$ . Set  $q_A = \text{End}(A, \Omega_R)$  and  $q_B = \text{End}(B, \Omega_R)$ . If  $q_A$  and  $q_B$  are at distance at least one on  $\Omega_R$  with respect to  $\text{Ends}(\mathcal{P}, \Omega_R)$ , set  $H = G(\Omega_L, \Omega_R) \cup A \cup B$ . Otherwise,  $q_A, q_B \in V(\Omega_R(i))$  for some  $0 \leq i \leq n-1$ . By (P3), there exists a path  $C \in \mathcal{Q}$  such that  $\text{End}(C, \Omega_R) \notin V(\Omega_R(i))$ . Set  $H = G(\Omega_L, \Omega_R) \cup A \cup B \cup C$ .

For each  $i \in \{0, \dots, n-1\}$ ,  $P_i$  intersect each of  $\Omega_L$  and  $\Omega_R$  in a single vertex. In addition, by definition,  $P'_i = P_i \cap G(\Lambda)$  is a sub-path of  $P_i$  with  $x_i \in V(P'_i)$ . Let  $H_1$  be obtained from  $H$  by contracting the path  $P'_i$  into the vertex  $x_i$ , for  $i = 0, \dots, n-1$ . By Theorem 4.1,  $H_1$  contains a  $K_6$  minor, and hence also  $G$ . This completes the proof.

## 6.2 Proof of Theorem 1.2 when $\Lambda$ is 1-sided

Since the deletion of any vertex decreases the non-separating face-width at most by 1, we may assume that  $\text{nsfw}(G) = 7$ . By Theorem 2.7, we may also assume that  $G$  is 3-connected and that  $\text{fw}(G) \geq 6$ . Moreover, we shall assume throughout this subsection that  $\Lambda$  is 1-sided. Let  $G'$  be the embedded graph obtained from  $G$  by cutting the surface along  $\Gamma(\Lambda)$  and capping off the resulting cuff with a disk  $F$ . In  $G'$ , every vertex  $x_i \in X(\Lambda)$  ( $i = 0, 1, \dots, 6$ ) is split into two copies,  $x'_i$  and  $x''_i$ , and the vertices  $x'_0, x'_1, \dots, x'_6, x''_0, x''_1, \dots, x''_6$  appear on the boundary of  $F$  in the listed cyclic order. By Theorem 1.1 we may assume that  $\Sigma$  is not the projective plane, thus the resulting surface  $\Sigma'$  is not the sphere. By Theorem 2.5,  $\text{nsfw}(G') \geq \lceil \frac{1}{2} \text{nsfw}(G) \rceil \geq 4$  and  $\text{fw}(G') \geq 3$ .

Let us first consider the possibility that  $\text{fw}(G') = 3$ . Let  $\Gamma'$  be the corresponding non-contractible curve. Clearly,  $\Gamma'$  involves the face  $F$  and two other faces  $A, B$  that are also faces of  $G$  in  $\Sigma$ . Let  $x, F, y, B, z, A, x$  be the corresponding closed face-chain in  $\Sigma'$ . The curve  $\Gamma'$  is surface-separating on  $\Sigma'$  since  $\text{nsfw}(G') \geq 4$ . We can view  $\Gamma'$  as a simple closed curve  $\Gamma_0$  in  $\Sigma$  by replacing the part of  $\Gamma'$  in  $F$  by a segment of  $\Gamma(\Lambda)$ . We can do this in two ways, so we may take a segment of  $\Gamma(\Lambda)$  such that  $\Gamma_0$  intersects at most 3 vertices in  $X(\Lambda)$  that are different from  $x$  and  $y$ . Since  $\text{nsfw}(G) \geq 7$ ,  $\Gamma_0$  is surface-separating in  $\Sigma$  (possibly contractible) and it separates the surface into two non-spherical surfaces  $\Sigma_1$  and  $\Sigma_2$ . One of them, say  $\Sigma_1$ , contains a 1-sided curve corresponding to  $\Lambda$ . The part of  $\Gamma'$  disjoint from the interior of the face  $F$  can be combined in  $\Sigma$  with two segments contained in  $\Lambda$  to give two closed curves. One of them is  $\Gamma_0$ , and we call the other one  $\Gamma_1$ . Observe that  $\Gamma_1$  is homologous to  $\Lambda$  (thus 1-sided) and  $\Gamma_0$  is surface-separating in  $\Sigma$ . Since  $\text{nsfw}(G) \geq 7$  and  $\text{fw}(G) \geq 6$ ,  $\Gamma_1$  necessarily passes through four consecutive vertices in  $X(\Lambda)$ , and  $\Gamma_0$  passes through the remaining three vertices in  $X(\Lambda)$ . We may assume that  $\Gamma_0$  passes through  $x_0, x_1, x_2$  and through the vertices  $x, y, z$ . One particular observation is that  $x, y \notin X(\Lambda)$  and that the face-chain of  $\Gamma_0$  is  $x, A, z, B, y, F_2, x_2, F_1, x_1, F_0, x_0, F_6, x$  (see Figure 5).

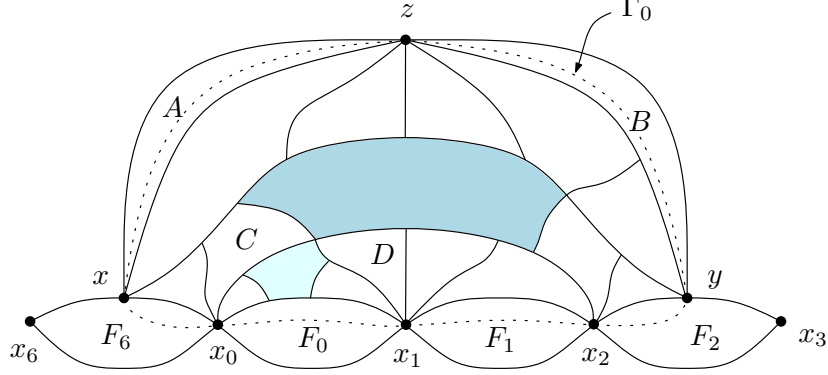


Figure 5: The surface  $\Sigma'_2$

Let us consider the set  $\mathcal{F}$  of faces of  $G$  that lie inside  $\Sigma_0$  and are incident with vertices in  $X(\Gamma_0)$ . Each such face is incident with precisely one vertex in  $X(\Gamma_0)$ . If not, we would either contradict 3-connectivity of  $G$  or the fact that  $\text{fw}(G) \geq 6$ . If a face  $Q \in \mathcal{F}$  is incident with  $t \in X(\Gamma_0)$ , we say that  $Q$  is a  $t$ -face, and we let  $\mathcal{F}_t$  denote the set of all  $t$ -faces in  $\mathcal{F}$ .

We say that two distinct vertices  $s, t \in X(\Gamma_0)$  are at *distance*  $d$  if their minimum face-distance in the closed face-chain of  $\Gamma_0$  is equal to  $d$ . Note that  $d \in \{1, 2, 3\}$ .

Suppose that  $s, t \in X(\Gamma_0)$  are at distance 3 and that  $A \in \mathcal{F}_s$ ,  $B \in \mathcal{F}_t$ . If  $A$  and  $B$  have a vertex  $v$  in common, then the face-chain  $s, A, v, B, t$  and the two face-subchains of  $\Gamma_0$  give rise to two closed face-chains in  $\Sigma$  of length 5, so they determine contractible closed walks. The 3-path-property implies that  $\Gamma_0$  is also contractible. This contradiction shows that  $A \cap B = \emptyset$ .

If  $s, t \in X(\Gamma_0)$  are at distance 2 and  $A \in \mathcal{F}_s$ ,  $B \in \mathcal{F}_t$  have a vertex  $v$  in common, then we similarly see that one of the face-chains in  $\Sigma$  obtained in the same way as above is of length 4, the other one of length 6. The first one determines a contractible curve in  $\Sigma$ . We can re-route  $\Gamma_0$  through  $v$ , thus making  $\Sigma'_2$  smaller. By repeating this process as long as necessary, we may assume that faces in  $\mathcal{F}_s$  and  $\mathcal{F}_t$  are disjoint whenever  $s$  and  $t$  are at distance 2.

If  $s$  and  $t$  are at distance 1 and two faces,  $C \in \mathcal{F}_s$  and  $D \in \mathcal{F}_t$ , have a vertex  $v$  in common (e.g. the faces  $C, D$  depicted in Figure 5), then there is a face-chain of length 3 through  $s, t, v$  and the two faces. The corresponding closed curve  $\Gamma$  in  $\Sigma$  is contractible, and we add all faces in the interior of  $\Gamma$  into  $\mathcal{F}$ . After doing this for all possible choices of  $s, t, C, D$ , we define  $\partial\mathcal{F}$  as the set of edges that belong to precisely one face in  $\mathcal{F}$  and do not belong to any of the faces of  $\Gamma_0$ . The properties stated in the preceding paragraphs imply that  $\partial\mathcal{F}$  is a simple cycle in  $G$  that is homotopic to  $\Gamma_0$ . (In Figure 5, this cycle is represented as the boundary of the darker shaded area. All faces in the lighter shaded area belong to  $\mathcal{F}$  and form a disk in  $\Sigma$ .) Now we delete all edges and vertices in  $\Sigma_2$  that do not belong to any of the faces in  $\mathcal{F}$  and cap off the cycle  $\partial\mathcal{F}$  by pasting a disk onto it. This gives rise to a subgraph  $G_1$  of  $G$  embedded into the capped surface  $\Sigma_1$ . It is easy to see by using the 3-path-property that  $\text{nsfw}(G_1) \geq 7$  since every surface non-separating face-chain

through the disk of  $\partial\mathcal{F}$  can be rerouted to use the face-chain of  $\Gamma_0$  without increasing its length. Since the genus decreases by the reduction from  $\Sigma$  to  $\Sigma_1$ , such a reduction can be made only a finite number of times, eventually yielding a case where  $\text{fw}(G') \geq 4$ .

From now on, we shall assume that  $\text{fw}(G') \geq 4$ . Let us apply Theorem 2.6 to the embedding of  $G'$  in  $\Sigma'$  and the face  $F$ . For  $i = 0, 1$ , let  $C_i = C_i(F)$  be the cycle as in Theorem 2.6. Since  $\text{fw}(G') \geq 4$ , these two cycles are contractible in  $\Sigma'$ .

The boundary of  $F$  is a cycle in  $G'$ . In  $G$ , it corresponds to a closed walk which intersects itself transversally when passing through the vertices in  $X(\Lambda)$ , but it does not cross itself on the surface. In this sense we view  $C_0$  as a closed walk in  $G$ . Theorem 2.6 assures that  $C_0$  is homotopic to  $C_1$ .

Consider the cycle  $C_1$  in  $G$ . Cutting  $G$  along  $C_1$  separates  $\Sigma$  into two components, one of which contains  $\Lambda$  and  $C_0$ . This surface is homeomorphic to the Möbius strip. By capping off the cuff (pasting a disk onto  $C_1$ ), we obtain a graph embedded into the projective plane  $\Sigma_1$ . We denote by  $F_1$  the face in  $\Sigma_1$  bounded by the cycle  $C_1$ . We also denote by  $\Sigma_2$  the other bordered surface obtained after cutting  $\Sigma$  along  $C_1$ .

Let  $v \in V(C_1)$ . Since  $G$  is 3-connected and the embedding of  $G$  in  $\Sigma$  has face-width more than 3, the facial neighborhood of  $v$  forms a disk on the surface that is bounded by a cycle  $N_v$ . This cycle contains a path  $P_v$  whose ends  $x, y$  are on  $C_1$  but all edges and other vertices on this path lie in  $\Sigma_2 \setminus C_1$ . Moreover,  $P_v$  can be selected so that the cycle  $Q_v$  consisting of  $P_v$  and the  $(x, y)$ -segment of  $C_1$  containing  $v$  is contractible in  $\Sigma_2$ , and the interior of  $Q_v$  contains all faces that are incident with  $v$  and are contained in  $\Sigma_2$ . (The proof of this fact is essentially the same as the main argument in the proof of Theorem 2.6; cf. [8] or [10].) If  $u, v \in V(C_1)$  and the ends of the paths  $P_v$  and  $P_u$  interlace on  $C_1$ , contractibility of the cycles  $Q_v$  and  $Q_u$  implies that  $P_v$  and  $P_u$  intersect. This property has the following consequence. Let  $H'$  be the minor of  $G \cap \Sigma_2$  obtained from  $P = \cup_{v \in V(C_1)} P_v \cup C_1$  by contracting all edges in  $P$  whose both ends are outside  $C_1$ . Then  $H'$  consists of  $C_1$  together with some chords of  $C_1$  and some vertices whose all neighbors lie on  $C_1$ . The interlacing property stated earlier implies that  $H'$  can be drawn in the disk so that  $C_1$  is on the boundary of the disk. By inserting this disk into the face  $F_1$  in  $\Sigma_1$  we obtain a minor  $G''$  of  $G$  that is embedded into the projective plane. It is easy to see that the face-width of  $G''$  is at least 4, and by Theorem 1.1,  $G''$  contains  $K_6$  as a minor. Since  $G''$  is a minor of  $G$ , we conclude that  $G$  has  $K_6$  minor. This completes the proof of Theorem 1.2.

## Acknowledgement

We are grateful to Lino Demasi who provided us with the list of all triangle-free graphs in the  $Y\Delta$ -class of the projective  $4 \times 4$  grid.

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